# Théorème de Cartan (part 1)

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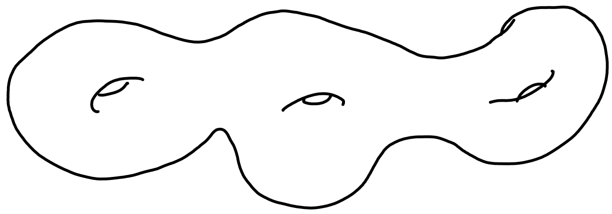
## I/ Overview

$S$ Riemann surface, ie complex 1-dim mfd



**Goal**: prove that its universal cover $\tilde{S}$ is $\mathbb{H}^2$ (or $\mathbb{S}^2$ or $\mathbb{R}^2$ if genus $\leq 1$).

Since $\tilde{S} \xrightarrow{\pi} S$ covering (with $\pi_1(\tilde{S}) = \{1\}$), $\tilde{S}$ is also a Riemann surface by lifting the charts.

We can endow $S$ with a conformal Riemannian metric $\rho$, ie

$$\forall x \in S \qquad \forall\ v, w \in T_x S \qquad \rho_x(iv, iw) = \rho_x(v, w)$$

($\Leftrightarrow$ $\rho_x$ is a multiple of the standard $\langle \cdot, \cdot \rangle$ in $\mathbb{C} \simeq \mathbb{R}^2$)

$(S, \rho)$ has a curvature $K_\rho$.

Idea: ① "flow" $\rho$ to get a conformal metric $\rho'$ such that $K_{\rho'} \equiv 1$

② use that $(\tilde{S}, \rho') = (\mathbb{H}^2, \rho_{hyp})$

In this lecture, we concentrate on the following

**Thm**: (Cartan)

Let $(M, \rho)$ be a smooth complete Riemannian manifold of dimension $n \geq 2$ such that $\pi_1(M) = \{1\}$ and constant sectional curvature equal to $c$.

If $c = -1$ then $M$ is isometric to $(\mathbb{H}^n, \rho_{hyp})$.

— $c = 0$ ——— $(\mathbb{R}^n, \rho_{eucl})$.

— $c = 1$ ——— $(\mathbb{S}^n, \rho_{eucl})$.

(Up to normalization these are the only possibilities.)

We will focus on the case $c = -1$, goal for $n = 2$ but true for all $n \geq 2$.

# II/ Crash course in Riemannian geometry

$M$ smooth manifold, $TM \to M$ tangent bundle, vector fields, differential....

rem: If $M \subset \mathbb{R}^n$ submanifold, vector fields on $M$ can always be smoothly extended, and properties we are interested in do not depend on the extension.

Convention: * $df_p : T_pM \to T_{f(p)}N$ differential (geometry)

* $Df_p = \left(\frac{\partial f_i}{\partial x_j}(p)\right)$ in $\mathbb{R}^n$ (calculus)

Derivation:

A vector field $X \in \Gamma(TM)$ is a derivation, that is an operator $C^\infty(M) \to C^\infty(M)$ which

① is $\mathbb{R}$-linear

② satisfies the Leibniz rule

Concretely, $X : f \mapsto df(X)$ is linear w.r.t $f$ and $X(fg) = g(X.f) + f(X.g)$.

Lie bracket: If $X, Y \in \Gamma(TM)$ then $f \mapsto X.(Y.f) - Y.(X.f)$ is a derivation

the Lie bracket $[X,Y]$ is the associated vector field.

prop: ① $[X,Y] \equiv 0 \iff$ the flows of $X$ and $Y$ commute

(ex: $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] \equiv 0$ in $\mathbb{R}^n$)

② If $\varphi$ is a diffeomorphism then $\varphi_*[X,Y] = [\varphi_* X, \varphi_* Y]$.

Riemannian metric: $g = (g_x)_{x\in M}$ smooth family of inner products on $(T_xM)_{x\in M}$

($\iff \forall X, Y \in \Gamma(TM)$ $x \mapsto g_x(X(x), Y(x))$ is smooth)

ex: 1) $M \subset \mathbb{R}^n$ and $g_x(u,v) = \langle u, v\rangle$ Euclidean (Euclidean metric)

2) $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ and $g_x(u,v) = \dfrac{\langle u,v\rangle}{x_n^2}$ (hyperbolic metric)

Length and energy of curves, $\gamma : [a,b] \to M$ $C^1$

the length of $\gamma$ is $L(\gamma) := \int_a^b \|\gamma'(t)\|\, dt = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt$

its energy is $E(\gamma) = \frac{1}{2}\int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t))\, dt$

By Cauchy-Schwarz $L(\gamma)^2 \leq 2(b-a)\, E(\gamma)$ with equality iff $\|\gamma'\|$ is constant.

Rem: $E(\gamma)$ depends on the parametrization, $L(\gamma)$ only depends on "$\gamma([a,b])$".

<u>Riemannian distance</u>: $d_\rho(p,q) = \mathrm{Inf}\{L(\gamma) \mid \gamma \text{ smooth path from } p \text{ to } q\}$ defines a distance on $M$.

On a manifold, no good way to compare vectors in $T_xM$ and $T_yM$.

<u>Connection</u>: An affine connection on $M$ is

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \quad \text{which} \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

1. is $\mathbb{R}$-linear w.r.t $Y$
2. is $C^\infty(M)$-linear w.r.t $X$
3. satisfies Leibniz rule $\quad \nabla_X(fY) = df(X)\,Y + f\,\nabla_X Y$

<u>ex</u>: 1) $D$ on $U \subset \mathbb{R}^n$.

2) for a submfd $M \subset \mathbb{R}^n$ $\quad \nabla^M_X Y = \mathrm{proj}^{\perp}_{T_xM}(D_X Y)$

$\boxed{\begin{array}{l} \textbf{Thm: } \text{Given } (M,\rho) \text{ Riemannian there exists a unique affine connection } \nabla \text{ such that} \\ \text{1. } \forall X,Y \in \Gamma(TM) \quad \nabla_X Y - \nabla_Y X = [X,Y] \quad \text{(torsion-free)} \\ \text{2. } \forall X,Y,Z \in \Gamma(TM) \quad d_X\,\rho(Y,Z) = \rho(\nabla_X Y, Z) + \rho(Y, \nabla_X Z) \quad \text{(metric connection)} \\ \text{It is called the Levi-Civita connection.} \end{array}}$

<u>Proof</u>: Assume $\nabla$ exists for any $X,Y,Z \in \Gamma(TM)$

$$\begin{aligned} X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y X, Z\rangle \\ &\quad + \langle X, \nabla_Y Z\rangle - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \\ &= \langle X, \underbrace{\nabla_Y Z - \nabla_Z Y}_{[Y,Z]}\rangle + \langle Y, \underbrace{\nabla_X Z - \nabla_Z X}_{[X,Z]}\rangle + \langle Z, \underbrace{\nabla_X Y - \nabla_Y X}_{[X,Y]}\rangle + 2\langle Z, \nabla_Y X\rangle \end{aligned}$$

Hence $\langle Z, \nabla_Y X\rangle = \frac{1}{2}\Big(X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle - \langle Z,[X,Y]\rangle\Big)$

i.e. $\nabla$ is completely determined by $\langle \cdot,\cdot\rangle$ and $[\cdot,\cdot]$.

Thus $\nabla$ is unique and we have the only candidate. Need to check that it works ...

## What are the critical points of the energy?

Let $\gamma : ]-\varepsilon,\varepsilon[\times[a,b] \longrightarrow M$ be a family of curves
$(s,t) \longmapsto \gamma_s(t)$

$$\frac{d}{ds}E(\gamma_s) = \int_a^b \underbrace{\frac{1}{2}\frac{d}{ds}\left(\left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t}\right\rangle\right)}_{} dt = \int_a^b \frac{1}{2}\, \frac{\partial \gamma}{\partial s}\cdot\left(\left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t}\right\rangle\right) dt$$

$$\frac{1}{2}\frac{d}{ds}\left(g_{\gamma(s,t)}\left(\frac{\partial \gamma}{\partial t}(s,t), \frac{\partial \gamma}{\partial t}(s,t)\right)\right) = d_{\frac{\partial\gamma}{\partial s}(s,t)}\left(\frac{1}{2}g\left(\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t}\right)\right)$$

$$= \int_a^b \left\langle \nabla_{\frac{\partial\gamma}{\partial s}}\left(\frac{\partial \gamma}{\partial t}\right), \frac{\partial \gamma}{\partial t}\right\rangle dt = \cdots \underset{IPP}{=} \int_a^b \left\langle \frac{\partial \gamma}{\partial s}, \nabla_{\frac{\partial\gamma}{\partial t}}\left(\frac{\partial \gamma}{\partial t}\right)\right\rangle dt.$$

Hence $\gamma : [a,b] \to M$ is a critical point of $E$
$\iff \forall\, V$ vector field along $\gamma$ $\int_a^b \langle V, \nabla_{\dot\gamma}\dot\gamma\rangle\, dt = 0$
$\iff \nabla_{\dot\gamma}\dot\gamma \equiv 0.$

Critical points of $E$ are the curves such that $\nabla_{\dot\gamma}\dot\gamma \equiv 0$. They are called <u>geodesics</u>.

<u>déf</u>: A vector field $X$ is said to be <u>parallel along</u> $\gamma$ if $\nabla_{\dot\gamma}X \equiv 0$.

<u>prop</u>: $\forall v \in T_{\gamma(0)}M$, there exists a unique parallel vector field $X$ along $\gamma$ such that $X(\gamma(0)) = v$.

<u>proof</u>: Cauchy–Lipschitz.

<u>Thm</u>: $\forall x \in M$ $\forall v \in T_xM$ $\exists!$ geodesic $\gamma : I_\gamma \to M$ such that $\gamma(0) = x$ $\dot\gamma(0) = v$.

<u>proof</u>: Cauchy–Lipschitz for $2^{nd}$ order ODE.

<u>def</u>. $(M,g)$ is <u>(geodesically) complete</u> if all geodesics are defined on $\mathbb{R}$.

<u>Thm</u>: (Hopf–Rinow) The following are equivalent (if $M$ is connected)

① $(M,g)$ is geodesically complete
② $(M, d_g)$ is a complete metric space
③ $\exists p \in M$ such that every geodesic starting at $p$ is defined on $\mathbb{R}$.

<u>Rem</u>: Before this theorem, we never used that $g_x$ is positive-definite, only non-degenerate.