# The Ricci flow on surfaces

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# Foreword

I prepared this set of notes for a series of 5 lectures I gave at the CIMPA school *“On Geometric Flows”* held at *Jadavpur University* in Kolkata in December 2016.

The goal is to present the Ricci flow on surfaces. The main interest of this topic is to give a lot of the leading ideas needed to understand Ricci flow in general while keeping the technical level not too high for beginners. The main focus of these notes will be to give a proof of the uniformization theorem using the Ricci flow.

We will start with a short primer on the Riemannian geometry of surface, using Cartan’s method of moving frames. This approach has been chosen because it allows to compute the evolution of geometric quantities along the Ricci with minimal effort.

Then we will move on to Ricci flow per se, discussing examples, existence and uniqueness and a few preliminary results.

The next chapter will briefly present the uniformization theorem, and explain why it is enough to show convergence of the Ricci flow to a constant curvature metric to prove it.

We will then prove convergence of the Ricci flow when the genus of the underlying surface is greater than 1. This was first done by Hamilton in 86.

The final chapter gives new tools to prove convergence for the genus 0 case, that is when the underlying surface is the sphere. This is actually much more difficult, and a proof of convergence of the Ricci flow which doesn’t use the uniformization theorem has been found only quite recently. The ideas we present come from a paper by Andrews and Bryan, and relies on isoperimetric inequalities.

References are :

- For the first chapter, one can find a similar treatment in *Differential forms and applications* by Do Carmo.

- For chapters 2 to 4, my main source was *The Ricci flow, an introduction* by Chow an Knopff. The treatment of the Ricci flow of surfaces is located in chapter 5 of this book. One can also have a look at Hamilton’s paper *The Ricci flow on surfaces.* The interlude on the link between uniformization of Riemannian surfaces, Riemann surfaces and algebraic curves is taken from *Uniformisation des surfaces de Riemann, Retour sur un théorème centenaire* by Henri Paul de Saint-Gervais (a placeholder name for a group of french mathematicians).

- The final chapter is based on the article *Curvature Bounds By Isoperimetric Comparison For Normalized Ricci Flow On The Two-sphere* by Andrews and Bryan.

Please feel free to report inaccuracies, misspellings, confusing statements, errors at :
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# Contents

# 1 The curvature of surfaces using Cartan's Method

We will briefly develop Cartan's movig frame method for the study of the curvature of surfaces. It allows to define the curvature in terms of the exteriror differential of differential forms, rather than in terms of covariant derivative of vector fields.

In our context, the main interest of this approach is that the exterior derivative depends only on the smooth structure while the covariant derivative depends on the Riemannian metric. When the Riemannian metric itself is moving with repsect to a time parameter $t$, the covariant derivative and the time derivative will not commute, while the exterior differential will commute with the time derivative.

## 1.1 Differential forms survival kit

If $X$ is a vector field and $f$ a smooth function, $X \cdot f$ will denote the derivative of $f$ in the direction of $X$, which is equal to the smooth function $df(X)$.

$[X, Y]$ will denote the Lie bracket of two vector fields.

**Definition 1.1.1.** *A* 1*-form on a surface $M^2$ is a linear map $\alpha$ from the space of smooth vector fields on $M^2$ to $C^\infty(M)$ such that $\alpha(fX) = f\alpha(X)$ for any $f \in C^\infty(M)$.*

*Example* 1.1.2. The differential $df$ of a function is a 1-form. On $\mathbb{R}^2$ the 1-forms $dx$ and $dy$ are the differentials of $(x, y) \mapsto x$ and $(x, y) \mapsto y$.

*Example* 1.1.3. On $\mathbb{R}^2$ any 1-form can written as $\alpha = \alpha_x(x, y)dx + \alpha_y(x, y)dy$. If $V = V_x(x, y)\partial_x + V_y(x, y)\partial_y$, then $\alpha(V) = \alpha_x V_x + \alpha_y V_y$.

*Exercise* 1.1.4. Show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth, then $df = \partial_x f dx + \partial_y f dy$.

**Definition 1.1.5.** *A* 2*-form on a surface $M^2$ is an antisymmetric bilinear map $\eta$ from the space of smooth vector fields on $M^2$ to $C^\infty(M)$ such that $\eta(fX, Y) = f\eta(X, Y)$ for any $f \in C^\infty(M)$.*

On a dimension 2 vector space, there is only one antisymmetric bilinear form up to scaling. Assume one can find a nowhere vanishing 2-form $\eta$ on $M^2$ (this is equivalent to requiring $M^2$ to be orientable), then any 2-form $\tilde{\eta}$ can be written $f\eta$ for some smooth function $f$.

The exterior product of two 1-forms $\alpha$ and $\beta$ is the two form given by :

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \beta(X)\alpha(Y).$$

The exterior differential of a 1-form is the 2-form defined by :

$$d\alpha(X, Y) = X \cdot (\alpha(Y)) - Y \cdot (\alpha(X)) - \alpha([X, Y]).$$

It satisfies that $d(f\alpha) = f d\alpha + df \wedge \alpha$ and $d(df) = 0$ when $f$ is a smooth function. Moreover, on a simply connected domain, the condition $d\alpha = 0$ is the only obstruction to the existence of a function $f$ such that $\alpha = df$.

There is a well defined notion of integral of 2-forms on an oriented manifold, denoted by $\int_M \eta$. To see this let us first assume that $\eta$ is a compactly supported 2-form on $\mathbb{R}^2$. Then $\eta = f(x,y)dx \wedge dy$ for some smooth function $f$. We set $\int_M \eta = \int f(x,y)\, dx\, dy$. Now consider an orientation preserving diffeomorphism $\varphi = (\varphi^1, \varphi^2)$ from $\mathbb{R}^2$ to itself. Then :

$$\begin{aligned}
\int_{\mathbb{R}^2} \varphi^* \eta &= \int_{\mathbb{R}^2} f(\varphi^1(x,y), \varphi^2(x,y)) d\varphi^1(x,y) \wedge d\varphi^2(x,y) \\
&= \int_{\mathbb{R}^2} f(\varphi^1(x,y), \varphi^2(x,y))(\partial_x \varphi^1 dx + \partial_y \varphi^1 dy) \wedge (\partial_x \varphi^2 dx + \partial_y \varphi^2 dy) \\
&= \int_{\mathbb{R}^2} f(\varphi^1(x,y), \varphi^2(x,y))(\partial_x \varphi^1 \partial_y \varphi^2 - \partial_y \varphi^1 \partial_x \varphi^2) dx \wedge dy \\
&= \int_{\mathbb{R}^2} f(\varphi(x,y)) \det D\varphi(x,y) dx \wedge dy \\
&= \int_{\mathbb{R}^2} f(\varphi(x,y)) \det D\varphi(x,y) dx\, dy \\
&= \int_{\mathbb{R}^2} f(u,v)\, du\, dv \\
&= \int_{\mathbb{R}^2} f(u,v) du \wedge dv \\
&= \int_{\mathbb{R}^2} \eta
\end{aligned}$$

Once we avec established this diffeomorphism invariance, we can use it to define the integral of a 2-form on a surface using a partition of unity subordinated to a finite cover by coordinate charts.

A special case of Stokes theorem tells that : $\int_M d\alpha = 0$ for any 1-form whenever $M$ is compact.

## 1.2 Moving frames on surfaces and the connection 1-form

Let $M^2$ be a smooth surface and $g$ be a smooth Riemannian metric on $M^2$. We will also assume that $M^2$ is oriented and by orthonormal basis, we will mean oriented orthonormal basis.

**Proposition 1.2.1.** *In every coordinate patch one can build a smooth frame $(E_1, E_2)$ which is orthonormal at each point.*

*Proof.* Let $\varphi : \Omega \to M$ be a coordinate patch. By definition of a coordinate patch, the vector fields $\partial_x \varphi$ and $\partial_y \varphi$ on $u(\Omega)$ are linearly independant. Set :

$$E_1 = \frac{\partial_x \varphi}{\|\partial_x \varphi\|_g}, \ E_2 = \frac{\partial_y \varphi - g(\partial_y \varphi, E_1)E_1}{\|\partial_y \varphi - g(\partial_y \varphi, E_1)E_1\|_g}.$$

□

When one has a frame $E_1, E_2$, one can define a dual coframe of 1-forms $(\eta^1, \eta^2)$ by requiring that $\eta^i(E_j) = \delta^i_j$ at each point.

*Remark* 1.2.2. **Important warning :** An orthonormal coframe, while really convenient for the calculations, is never unique. This means that one need to be careful when defining new objects from the coframe that these object do not depend on the coframe.

Also a coframe is almost always only defined locally (it comes from a pair of nowhere vanishing vector field, the "Hairy ball" theorem tells us that no such vector filed exist on $\mathbb{S}^2$). So that one has to be careful when trying to prove global result using the coframe method.

Once the coframe is defined, the metric takes a quite nice form, one can write :

$$g = (\eta^1)^2 + (\eta^2)^2$$

*Example* 1.2.3. Let us look at the Euclidean metric in polar coordinates : $dr^2 + r^2 d\theta^2$ to find an orthonormal coframe, we need to write the metric as a "sum of squares". On this example, this is easily done :

$$dr^2 + r^2(d\theta)^2 = (dr)^2 + (r\, d\theta)^2$$

so that an orthonormal coframe is $(dr, r\, d\theta)$.

*Exercise* 1.2.4. Show that for any one form $\alpha$ one has $\alpha = \alpha(E_1)\eta^1 + \alpha(E_2)\eta^2$, show that for any vector field $X$ one has $X = \eta^1(X)E_1 + \eta^2(X)E_2$.

Since $(E_1, E_2)$ is orthonormal, we have :

$$\eta^1(X) = g(E_1, X),\ \eta^2(X) = g(E_2, X).$$

*Exercise* 1.2.5. Show that if $[E_1, E_2] = 0$, then $d\eta^1 = d\eta^2 = 0$. From this fact, show that there is (at least locally) a coordinate chart $\varphi : \Omega \to M^2$ such that $E_1 = \partial_x \varphi$, $E_2 = \partial_y \varphi$ and that $\varphi$ is a Riemannian isometry.

Just as the volume form of $\mathbb{R}^2$ is $dx \wedge dy$, the Riemannian volume form of $(M^2, g)$ is given by :

$$dv_g = \theta^1 \wedge \theta^2.$$

The fact that $(E_1, E_2)$ is orthonormal allows certain simplifications in the computation of covariant derivatives.

**Proposition 1.2.6.** *For every tangent vector field, set $\omega(X) = g(D_X E_1, E_2)$ then :*

- *$\omega$ is a one form.*
- *$D_X E_1 = \omega(X) E_2$*
- *$D_X E_2 = -\omega(X) E_1$.*

*Proof.* The fact that $X \mapsto D_X Y$ is tensorial implies that $\omega$ is a 1-form.

Then since $g(E_1, E_1) = 1$ :

$$g(D_X E_1, E_1) = \tfrac{1}{2} X \cdot g(E_1, E_1) = 0.$$

Thus, $D_X E_1$ is proportional to $E_2$ and :

$$D_X E_1 = g(D_X E_1, E_2) E_2 = \omega(X) E_2.$$

The third formula follows using that :

$$g(D_X E_1, E_2) + g(E_1, D_X E_2) = X \cdot g(E_1, E_2) = 0.$$

□

**Definition 1.2.7.** *$\omega$ is called the connexion form of the frame $(E_1, E_2)$.*

*Exercise* 1.2.8. Show that

$$D_X Y = (X \cdot \eta^1(Y))E_1 + (X \cdot \eta^2(Y))E_2 + \omega(X)(\eta^1(Y)E_2 - \eta^2(Y)E_1).$$

## 1.3 Cartan's structure equations

The first structure equation shows how one can compute the exterior differential of $\eta^1$ and $\eta^2$ using $\omega$.

**Proposition 1.3.1.** *The 1-form $\omega$ is the **unique** 1-form which satisfies :*

$$\begin{cases} d\eta^1 = \omega \wedge \eta^2 \\ d\eta^2 = -\omega \wedge \eta^1. \end{cases}$$

*Proof.* We use the intrinsic definition of the exterior differential, together with the defining properties of the Levi-Civita connection :

$$\begin{aligned} d\eta^1(X,Y) =& X \cdot \eta^1(Y) - Y \cdot \eta^1(X) \\ =& X \cdot g(E_1, Y) - Y \cdot g(E_1, X) - Y \cdot g(E_1, [X,Y]) \\ =& g(D_X E_1, Y) + g(E_1, D_X Y) - g(D_Y E_1, X) - g(E_1, D_X Y) - g(E_1, [X,Y]). \end{aligned}$$

We now use that $D_V E_1 = \omega(V)E_2$ and that $D_X Y - D_Y X = [X,Y]$ to get :

$$\begin{aligned} d\eta^1(X,Y) =& \omega(X) g(E_2, Y) - \omega(Y) g(E_2, X) \\ =& \omega(X)\eta^2(Y) - \omega(Y)\eta^2(X) \\ =& \omega \wedge \eta^2(X,Y) \end{aligned}$$

which proves the first formula, the proof of the second formula is similar. □

The formula in itself is nice for calculations, but the really interesting part is that these two equations characterize $\omega$, this gives a way to compute $\omega$ without knowing anything about covariant derivatives. It is actually possible to prove the existence of $\omega$ without any mention of covariant derivatives.

*Example* 1.3.2. Recall that $(dr, r\,d\theta)$ is an orthonormal coframe for the metric $dr^2 + r^2(d\theta)^2$. The first structure equation tells us that :

$$\begin{cases} d(dr) = \omega \wedge (r\,d\theta) \\ d(r\,d\theta) = -\omega \wedge dr \end{cases}$$

Since $d(dr) = 0$ and $d(r\,d\theta) = dr \wedge d\theta$, one can guess that $\omega = d\theta$.

*Exercise* 1.3.3. Show that $\omega = 0$ if and only there exist two function $p \in M \mapsto x(p)$ and $p \in M \mapsto y(p)$ such that $p \in M \mapsto (x(p), y(p)) \in \mathbb{R}^2$ is an isometry.

The second structure equation allows to compute the curvature from the connection form :

**Proposition 1.3.4.**

$$d\omega = -K_g\, \eta^1 \wedge \eta^2$$

*where $K_g$ is the sectionnal (or Gaussian) curvature of $(M^2, g)$.*

*Proof.* Since $d\omega$ is a 2-form and $\eta^1 \wedge \eta^2$ is a volume form, one can always find a function $f$ such that $d\omega = f\, \eta^1 \wedge \eta^2$. To prove that $f = -K_g$, it is enough to show that $d\omega(E_1, E_2) = -K_g$. We thus compute :

$$\begin{aligned} d\omega(E_1, E_2) =& E_1 \cdot \omega(E_2) - E_2 \cdot \omega(E_1) - \omega([E_1, E_2]) \\ =& E_1 \cdot g(D_{E_2} E_1, E_2) - E_2 \cdot g(D_{E_1} E_1, E_2) - g(D_{[E_1,E_2]} E_1, E_2) \\ =& g(D_{E_1} D_{E_2} E_1, E_2) + g(D_{E_2} E_1, D_{E_1} E_2) \\ & - g(D_{E_2} D_{E_1} E_1, E_2) - g(D_{E_1} E_1, D_{E_2} E_2) \\ & - g(D_{[E_1,E_2]} E_1, E_2) \end{aligned}$$

Using that $D_X E_i$ is orthogonal to $E_i$, we get that the two terms on the right are equal to 0. The terms on the left add up to give, by definition of th Riemann curvature tensor:

$$d\omega(E_1, E_2) = -R(E_1, E_2, E_1, E_2) = -K_g$$

by definition of the sectionnal curvature. $\square$

*Exercise* 1.3.5. If $(\eta^1, \eta^2)$ is an orthonormal coframe for the metric $g$, find an orthonormal coframe for $\lambda g$ ($\lambda$ is a constant) and use it to find the relation between the curvature of $\lambda g$ to the curvature of $g$.

*Exercise* 1.3.6. Consider the metric $g = \frac{dx^2+dy^2}{y^2}$. Show $E_1 = y\partial_x$, $E_2 = y\partial_y$) is an orthonormal frame. Build the associated coframe $(\eta^1, \eta^2)$ and show that the connection form satisfies $\omega = \eta^1$. Use this to show that $g$ has constant curvature $-1$.

*Exercise* 1.3.7. Consider a metric of the form $g = dr^2 + f(r)^2 d\theta^2$, build a simple orthonormal frame and the associated co-frame on it, and use it to compute the connection form and the curvature.

*Exercise* 1.3.8. Compute curvature of a diagonal metric $g = \varphi(x, y)^2 dx^2 + \psi(x, y)^2 dy^2$.

The formula $d\omega = -K_g \eta^1 \wedge \eta^2$ suggests that $\int_M K_g dv_g$ should have some special properties. Actually Cartan's second structure equation allows to give a proof of the Gauss-Bonnet theorem :

**Theorem 1.3.9.** *Let $(M^2, g)$ be a compact Riemannian surface, then :*

$$\int_{M^2} K_g dv_g = 2\pi\chi(M^2).$$

*Remark* 1.3.10. $\chi(M^2)$ is the Euler characteristic of the surface $M^2$, it is a topological invariant. In 2d it can be defined as $2 - 2g$ where $g$ is the numer of "holes" in the surface, called the . For instance :

- if $M^2$ is a sphere, then $g = 0$ and $\chi(M^2) = 2$.
- if $M^2$ is a torus, then $g = 1$ and $\chi(M^2) = 0$.
- more generally, if $M^2$ is a surface with $g \geq 2$ holes, $\chi(M^2) = 2 - 2g \leq -2$.

In the next exercise we prove a special case.

*Exercise* 1.3.11. Assume $M^2$ admits a nowhere vanishing vector field $X$. Show that for any Riemannian metric $g$ on $M^2$ :

$$\int_{M^2} K_g dv_g = 0.$$

*The existence of a nowhere vanishing vector field is equivalent to the vanishing of the Euler characteristic $\chi(M)$, for orientable surfaces, only the torus carries such a vector field.*

*Exercise* 1.3.12. For a smooth function $\gamma$ set :

$$\begin{cases} \tilde{\eta}^1 = \cos\gamma\,\eta^1 - \sin\gamma\,\eta^2 \\ \tilde{\eta}^2 = \sin\gamma\,\eta^1 + \cos\gamma\,\eta^2 \end{cases}$$

Show that $(\tilde{\eta}^1, \tilde{\eta}^2)$ is an orthonormal coframe, denote by $\tilde{\omega}$ its connexion form. Show that $\tilde{\omega} = \omega + d\gamma$.

Show that if $K_g = 0$, one can find $\alpha$ such that $\tilde{\omega} = 0$. Deduce from this that $(M^2, g)$ is locally isometric to $\mathbb{R}^2$ if and only if $K_g = 0$.

## 1.4 The Laplacian and the Hodge star

The Laplacian on a Riemannian manifold has several equivalent definitions.

**Definition 1.4.1.** *The Hessian of a smooth function $f : M \to \mathbb{R}$ with respect to the metric $g$ is the 2-tensor $Ddf$.*

*Exercise* 1.4.2. Show that $Ddf(X,Y) = Ddf(Y,X)$ and that $Ddf(X,Y) = g(D_X \nabla f, Y)$.

*Exercise* 1.4.3. Show that if $X$ is a vector field, then $Ddf(X,X) = X \cdot (X \cdot f) - (D_X X) \cdot f$.

**Definition 1.4.4.** *The Laplacian of $f : M^2 \to \mathbb{R}$ is the trace of the hessian :*

$$\Delta_g f = Ddf(E_1, E_1) + Ddf(E_2, E_2)$$

*Exercise* 1.4.5. Show that :

$$\Delta f\, \eta^1 \wedge \eta^2 = (E_1 \cdot (E_1 \cdot f) + E_2 \cdot (E_2 \cdot f))\, \eta^1 \wedge \eta^2 - \omega \wedge df.$$

And that if $\alpha = -(E_2 \cdot f)\theta^1 + (E_1 \cdot f)\theta^2$, then $d\alpha = \Delta f \eta^1 \wedge \eta^2$.

Let $*_g$ denote the $C^\infty(M)$ linear operator on 1-forms which sends $\eta^1$ to $\eta^2$ and $\eta^2$ to $-\eta^1$ (it is a $\pi/2$ rotation in the space of 1-forms). Then :

$$*_g df = *_g \left((E_1 \cdot u)\eta^1 + \cdot(E_2 \cdot u)\eta^2\right) = -(E_2 \cdot f)\eta^1 + (E_1 \cdot f)\eta^2.$$

In the previous exercise we thus have proved that $\Delta_g f\, \eta^1 \wedge \eta^2 = d * du$.

*Exercise* 1.4.6. Show that if $\tilde{g} = e^{2u}g$, $*_{\tilde{g}} = *_g$. Deduce from this that $\Delta_{\tilde{g}} f = e^{-2u}\Delta_g f$.

*Exercise* 1.4.7. Use Stokes theorem to show that $\int_M \Delta_g u dv_g = 0$.

*Exercise* 1.4.8. Show that $du \wedge *dv = g(\nabla u, \nabla v) dv_g$. Deduce from this that :

$$\int_M (\Delta_g u) v \, dv_g = -\int_M g(\nabla u, \nabla v) dv_g.$$

*Exercise* 1.4.9. Show that if $f$ has a local minimum at $p \in M^2$, then $\Delta f(p) \geq 0$.

## 1.5 Time derivative of the curvature along a conformal change.

Consider a one parameter family of Riemannian metrics $g(t)$ on $M^2$ such that :

$$\frac{\partial}{\partial t} g = 2f\, g$$

for some smooth function $f$. The objective of this section is to find a nice expression for the time derivative of $K_g$.

We will try to build frames and coframes adapted to the problem. Assume $(E_1^0, E_2^0)$ is an orthonormal frame at $t = 0$, then if $\dot{E}_i = -f E_i$ and $E_i(0) = E_i^0$ we have that :

$$\begin{aligned} \dot{\overbrace{g(E_i, E_j)}} &= \dot{g}(E_i, E_j) + g(\dot{E}_i, E_j) + g(E_i, \dot{E}_j) \\ &= 2f\, g(E_i, E_j) - f\, g(E_i, E_j) - f\, g(E_i, E_j) = 0. \end{aligned}$$

So $E_i$ is a (time dependent) orthonormal frame. Differentiating the relation $\eta^i(E_j) = \delta^i_j$, we get that the associated coframe evolves by $\dot{\eta}^i = f\, \eta^i$.

*Exercise* 1.5.1. Recall that the gradient $\nabla^g u$ of a smooth function $u : M^2 \to \mathbb{R}$ satisfies $du(X) = g(\nabla u, X)$ for every vector field $X$. Show that $\nabla^g u = (E_1 \cdot u)E_1 + (E_2 \cdot u)E_2$, and use it to compute the time derivative of $\nabla^{g(t)} u$.

From this we can easily get the time derivative of the volume form :

**Proposition 1.5.2.** *If $g(t)$ is a one parameter family of metrics such that $\frac{\partial}{\partial t} g = 2f\, g$, then :*

$$\frac{\partial}{\partial t} dv_{g(t)} = 2f\, dv_{g(t)}$$

*Proof.* We know that locally $dv_{g(t)} = \eta^1 \wedge \eta^2$, thus :

$$\dot{\overbrace{\eta^1 \wedge \eta^2}} = \dot{\eta}^1 \wedge \eta^2 + \eta^1 \wedge \dot{\eta}^2 = 2f\, \eta^1 \wedge \eta^2.$$

□

*Exercise* 1.5.3. Show that if $u : M \to \mathbb{R}$ is a smooth function, then :

$$\frac{d}{dt} \Delta_{g(t)} u = -2f \Delta_{g(t)} u.$$

We now want to take the time derivative of Cartan's first structure equation :

$$\begin{cases} d\eta^1 = \omega \wedge \eta^2 \\ d\eta^2 = -\omega \wedge \eta^1 \end{cases}$$

which gives, since time derivative and exterior differential commute :

$$\begin{cases} d\dot{\eta}^1 = \dot{\omega} \wedge \eta^2 + \omega \wedge \dot{\eta}^2 \\ d\dot{\eta}^2 = -\dot{\omega} \wedge \eta^1 - \omega \wedge \dot{\eta}^1 \end{cases}$$

Hence, since $\dot{\eta}^i = f\eta^i$ :

$$\begin{cases} df \wedge \eta^1 + f d\eta^1 = \dot{\omega} \wedge \eta^2 + f\omega \wedge \eta^2 \\ df \wedge \eta^2 + f d\eta^2 = -\dot{\omega} \wedge \eta^1 - f\omega \wedge \eta^1 \end{cases}$$

The Cartan structure equation then implies :

$$\begin{cases} df \wedge \eta^1 = \dot{\omega} \wedge \eta^2 \\ df \wedge \eta^2 = -\dot{\omega} \wedge \eta^1 \end{cases}$$

Now writing $df = (E_1 \cdot f)\eta^1 + (E_2 \cdot f)\eta^2$ and soving for $\dot{\omega}$ we have :

$$\dot{\omega} = -(E_2 \cdot f)\eta^1 + (E_1 \cdot f)\eta^2 = *df.$$

We now differentiate Cartan's second structure equation $d\omega = -K_g\eta^1 \wedge \eta^2$ to get :

$$d\dot{\omega} = -\dot{K}_g\eta^1 \wedge \eta^2 - 2fK_g\eta^1 \wedge \eta^2.$$

All that remains to be done to find $\dot{K}_g$ is to compute $d\dot{\omega}$. With the aid of Cartan's first structure equation we get :

$$\begin{aligned} d\dot{\omega} =& d * df \\ =& \Delta_g f \, \eta^1 \wedge \eta^2. \end{aligned}$$

Hence we have proved that :

**Proposition 1.5.4.** *If $g(t)$ is a one parameter family of metrics such that $\frac{\partial}{\partial t} g = 2f\, g$, then :*

$$\frac{\partial}{\partial t} K_{g(t)} = -\Delta_{g(t)} f - 2fK_{g(t)}.$$

*Exercise* 1.5.5. Let $u$ be a time dependent function $[0, T) \times M^2 \to \mathbb{R}$. Show that :

$$\frac{\partial}{\partial t}(\Delta_{g(t)} u) = \Delta_{g(t)} \left( \frac{\partial u}{\partial t} \right) - 2f\Delta_{g(t)} u.$$

*Exercise* 1.5.6. Use the same method to prove that if $\tilde{g} = e^{2u} g$ where $g$ is a metric and $u$ is a smooth function, then :

$$\tilde{\omega} = \omega - *du$$

and

$$e^{2u} K_{\tilde{g}} = K_g - \Delta_g u.$$

Deduce from this that $\int_M K_g dv_g = \int_M K_{\tilde{g}} dv_{\tilde{g}}$.

## 1.6 Existence of isothermal coordinates

Isothermal coordinates are $p \in M^2 \mapsto (x(p), y(p))$ such that the metric can be written in these coordinates as $e^{2u(x,y)}(dx^2 + dy^2)$. In a more precise way, if one considers $\varphi : (x, y) \mapsto p(x, y) \in M^2$ then $\varphi^* g = e^{2u}(dx^2 + dy^2)$.

$\varphi$ is said to be conformal, the length are distorted by a factor $e^{2u}$, but the measure of angle between intersectin curves is preserved.

**Theorem 1.6.1.** *Every point $p$ in a riemannian surface $(M^2, g)$ has a neighborhood $U$ such that there exist a map : $(x, y) \in V \subset \mathbb{R}^2 \mapsto U$ with $\varphi^* g = e^{2u}(dx^2 + dy^2)$.*

*Proof.* Consider a chart $U$ around $p$ where a local orthonormal coframe is defined, we can assume that $U$ is diffeomorphic to an open disk and that $\bar{U}$ is homeomorphic to a closed disk.

Let $v : U \to \mathbb{R}$ be a solution of the Poisson equation $\Delta_g v = K_g$ on $U$ (for instance with dirichlet boundary conditions).

Set $\tilde{g} = e^{2u} g$ on $U$. The curvature of $\tilde{g}$ can then be computed by the formula :

$$K_{\tilde{g}} = e^{-2u}(K_g - \Delta_g v) = 0.$$

Thus $(U, \tilde{g})$ is flat. We thus can find an isometry (after reducing $U$ if needed) :

$$\varphi : V \subset (\mathbb{R}^2, dx^2 + dy^2) \mapsto (U, \tilde{g}).$$

Now $\varphi^* \tilde{g} = dx^2 + dy^2$, so that :

$$\begin{aligned} \varphi^* g &= \varphi^*(e^{-2v} \tilde{g}) \\ &= e^{-2(v \circ \varphi)} \varphi^* \tilde{g} \\ &= e^{2u}(dx^2 + dy^2) \end{aligned}$$

with $u = -v \circ \varphi$. □

# 2 Basics of surface Ricci flow

On a Riemannian surface $(M^2, g)$, the Ricci curvature can be expressed as $\mathrm{Ric}_g = K_g g$ where $K_g$ is the Gauss curvature of the surface. Thus the Ricci flow which is defined in arbitrary dimension by $\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}_{g(t)}$ reduces to :

$$\frac{\partial g}{\partial t} = -2K_{g(t)}g$$

The results of the previous chapter then implies that :

**Proposition 2.0.2.** *If $g(t)$ is a solution to the Ricci flow on a surface then :*

$$\frac{d}{dt}dv_{g(t)} = -2K_{g(t)}dv_{g(t)}.$$

**Proposition 2.0.3.** *If $g(t)$ is a solution to the Ricci flow on a surface then :*

$$\frac{d}{dt}K_{g(t)} = \Delta_{g(t)}K_{g(t)} + 2K^2_{g(t)}.$$

## 2.1 First examples

Consider a surface $(M^2, g_0)$ of constant curvature $K_{g_0} = \kappa$. Then $g(t) = (1 - 2\kappa t)g_0$ is a solution to the Ricci flow. Indeed, $\frac{\partial}{\partial t}g = -2\kappa g_0 = -\frac{2\kappa}{1-2\kappa t}g(t)$, so all we need to show is that :

$$\frac{\kappa}{1-2\kappa t} = K_{g(t)}$$

The relation between the curvature of two conformal metrics gives exactly this equality.

Note that the time interval on which these solutions make sense depnds on $\kappa$ :

- when $\kappa > 0$, $t \in (-\infty, 1/2\kappa)$ and the solution shrinks to a point as $t$ goes to its upper bound.
- when $\kappa = 0$, $t \in (-\infty, +\infty)$, and the metric does not change.
- when $\kappa > 0$, $t \in (1/2\kappa, +\infty)$ and the solution shrinks to a point as $t$ goes to its lower bound while as $t$ goes to $+\infty$, the metric expands.

This hints at how the sign of the curvature strongly affects the beahvior of the Ricci flow.

Let us also remark that although not quite constant, these solution only change by scaling. We will see in the next section how to modify the Ricci flow equation to make constant curvature metrics actual fixed points.

Let us consider another example, the cigar soliton. It is the one parameter family of metrics $g(t)$ on $\mathbb{R}^2$ written as :

$$g(t) = \frac{dx^2 + dy^2}{e^{4t} + r^2}$$

where $r^2 = x^2 + y^2$.

If we write it in polar coordinates, we get :

$$g(t) = \frac{dr^2 + r^2 d\theta^2}{e^{4t} + r^2}$$

When $r$ is going to infinity, this is asymptotic to :

$$\frac{dr^2 + r^2 d\theta^2}{r^2} = d(\log r)^2 + d\theta^2$$

which is a flat cylinder.

Let us write :

$$\frac{\partial}{\partial t} g(t) = -4 \frac{e^{4t}}{e^{4t} + r^2} g(t).$$

So we need to show that $K_{g(t)} = 2\frac{e^{4t}}{e^{4t}+r^2}$, the curvature formula for conformally related gives exactly this.

*Exercise* 2.1.1. The computation can also be carried after writing the metric in polar coordinates. Show that if $r = e^{2t}\sinh(\rho)$, then $g$ can be written as : $d\rho^2 + \tanh(\rho)^2 d\theta^2$.

The previous exercise shows that after a **time dependent** change of coordinates, the metrics are actually all isometric.This is a consequence of the fact the $g(t)$ is a Ricci soliton.

## 2.2 Ricci flow versus normalized Ricci flow

The Ricci flow equation in dimension 2 is :

$$\frac{\partial g}{\partial t} = -2K_{g(t)} g.$$

Let us compute the evolution of the area of $(M^2, g(t))$ :

**Proposition 2.2.1.** *If $(M^2, g(t))$ is a solution to the Ricci flow, then :*

$$\frac{d}{dt} \operatorname{area}(M^2, g(t)) = -4\pi\chi(M^2)$$

*Proof.* In the previous chapter, we saw that $\frac{d}{dt} dv_{g(t)} = 2f\, dv_{g(t)}$ when $\frac{d}{dt} g = 2f\, g$. Applyng this with $f = -K_g$, we get :

$$\begin{aligned} \frac{d}{dt} \operatorname{area}(M^2, g(t)) &= \frac{d}{dt} \int_M dv_{g(t)} \\ &= \int_M \frac{d}{dt} dv_{g(t)} \\ &= -2 \int_M K_{g(t)} dv_{g(t)} \\ &= -4\pi\chi(M^2) \end{aligned}$$

where the last equality follows from Gauss-Bonnet theorem. □

Integrating this equality we get :

$$\text{area}(M^2, g(t)) = \text{area}(M^2, g(0)) - 4\pi\chi(M)t.$$

*Exercise* 2.2.2. Assume $(M^2, g(t))_{t\in(a,b)}$ is a Ricci flow on a compact surface $M^2$ such that $a = -\infty$ (ancient solution) or $b = +\infty$ (immortal solution). In each case, what can be said about $\chi(M^2)$ ?

The fact that the solution disappears after some time $T_{max}$ when $\chi(M) > 0$ is a bit problematic. There are two ways of dealing with this, one can either work with the solution which becomes singular at $T_{max}$ and rescale it properly or try to modify the PDE bby a suitable change of variable so that this singulartime doesn't happen. This second approach is really effective in dimension 2.

We want to avoid the area of a solution $g(t)$ going to 0. Set :

$$\kappa_g = \frac{\int_M K_g dv_g}{\text{area}(M, g(t))}$$

And consider the equation :

$$\frac{\partial}{\partial t} g = 2(\kappa_g - K_g)g.$$

One shows that this new flow preserves the area and that one can recover the usual Ricci flow from it. This the normalized Ricci flow (NRF) on surfaces.

*Exercise* 2.2.3. Show that one can find $\phi(\tau)$ and $\psi(\tau)$ such that, if $(M, g(t))$ is a Ricci flow then $(M, \varphi(\tau)g(\psi(\tau))$ is a normalized Ricci flow.

The formula already tells us that the only time independent solutions are the constant curcature metrics.

## 2.3 Reduction to a scalar PDE

Quick inspection of the equation for the normalised Ricci flow suggests that the angle between two vectors doesn't change along the flow. In other words, all the metrics $g(t)$ are conformal to each other and can be written $g(t) = e^{2u(t)} g_0$ Actually, one can write down $u$ explicitly as : $u(t) = \int_0^t (\kappa - K_{g(s)})ds$, which hints at the fact that curvature bounds will play a big role in the study of convergence of the Ricci flow.

**Proposition 2.3.1.** *$g(t)$ is a solution to the Normalised Ricci flow if and only if $g(t) = e^{2u(t)} g_0$ and :*

$$\frac{\partial}{\partial t} u = e^{-2u}(\Delta_{g_0} u - K_{g_0}) + \kappa.$$

*Proof.* Differentiating $u(t) = \int_0^t (\kappa - K_{g(s)})ds$, we get that $\frac{\partial}{\partial t} u = \kappa - K_{g(t)}$. Since $e^{2u} K_g = K_{g_0} - \Delta_{g_0} u$, we get the identity we are looking for. □

In some cases, it can be convenient to write the evolution equation for $w(t) = \frac{1}{2} \log u(t)$, so that $g(t) = w(t) g_0$, we get :

$$\frac{\partial}{\partial t} w = \Delta_{g_0} \log w + 2(\kappa w - K_{g_0}).$$

In particular, the leading term $\Delta_{g_0} \log w$ is equal to div $\left(\frac{1}{w}\nabla w\right)$, we say that it is in divergence form. This help making sense of weak solutions for instance.

## 2.4 Existence and uniqueness

The existence and uniqueness theory in arbitrary dimensions requires quite a lot af work, mainly due to :

- The fact that the equation is geometric, and is thus invariant under coordinate changes, which prevents it from being striclty parabolic.
- The fact that we are dealing with a system of PDE.

In the end one get the following result :

**Theorem 2.4.1** (Hamilton, DeTurck)**.** *For any smooth initial metric $g_0$ on a compact manifold $M$, there exist $T > 0$ and a unique solution $(g(t))_{t\in[0,T)}$ to the Ricci flow such that $g(0) = g_0$ and either :*

- $T = +\infty$.
- $T < +\infty$ *and the curvature of* $g(t)$ *becomes unbounded as* $t \to T$.

*Remark* 2.4.2. The theorem stated above also holds for the normalized Ricci flow.

In dimension 2 however, the previous discussion has taken care of both of these issues, and the existence and uniqueness theory can be dealt with using quite standard tools from PDE theory. We will not give complete proofs but will try to give a rough idea of how one could proceed.

### 2.4.1 Existence

We will work directly with the scalar equation for $w$ :

$$\frac{\partial}{\partial t}w = \mathrm{div}_{g_0}\left(\frac{1}{w}\nabla w\right) + 2(\kappa w - K_{g_0}).$$

together with some initial condition $w_0 > 0$.

One possible strategy is to use the linear theory of parabolic equations to recast our problem into a fixed point problems, and then use an abstract fixed point theorem to prove existence. The method we outline here uses the Schauder fixed point theorem, but other choices would be possible.

The map $w \mapsto \Phi(w)$ is defined, for functions $w : M \times [0,T) \to \mathbb{R}$ such that $w(0) = w_0$ as the solution $v$ to the following linear Cauchy problem :

$$\begin{cases} \frac{\partial}{\partial t}v = \mathrm{div}_{g_0}\left(\frac{1}{w}\nabla v\right) + 2(\kappa v - K_{g_0}) \\ v(0) = w_0. \end{cases}$$

We have that $w$ is a solution to our initial nonlinear problem if and only if $\Phi(w) = w$. To prove the existence of a fixed point we would need to :

- Find a Banach space $X$ such that $\Phi$ maps $B \subset X$ to $X$ continuously, for $B$ a closed convex set. This in particular means that $\Phi(v)$ is at least as regular as $v$.

- Prove that $\Phi(B)$ is relatively compact. This is usually done by proving that $\Phi(v)$ has better regularity than $v$, and apply compact embeddings from functional analysis.

Both will use heavily the available estimates for linear parabolic equations of the form :

$$\partial_t f = \operatorname{div}(a\nabla f) + bf + h$$

Where $a$, $b$ $h$ are functions on $M \times [0,T)$. The structure of our problem hints us that these estimates should depend as little as possible on $a$.

This kind of argument is implemented in Taylor's book *Partial differential equations, vol. 3* in the Holder space setting.

### 2.4.2 Uniqueness

Uniqueness can be asserted with simple integral methods.

**Proposition 2.4.3.** *Assume $w_1 : (0,T] \times M \to \mathbb{R}$ and $w_2 : (0,T] \times M \to \mathbb{R}$ satisfy :*

$$\begin{cases} \dfrac{\partial}{\partial t} w = \Delta_{g_0} \log w + 2(\kappa w - K_{g_0}), \\ \lim\limits_{t\to 0} \displaystyle\int_M |w_1(t) - w_2(t)| dv_{g_0} = 0. \end{cases}$$

*Then $w_1 = w_2$ for all $t \in (0,T]$.*

*Proof.* See for instance my preprint *Canonical smoothing of compact Alexandrov surfaces via Ricci flow*, Proposition 3.4. ☐

## 2.5 The maximum principle and its first applications.

Let us look at the evolution of the curvature for the normalized Ricci flow :

$$\frac{\partial}{\partial t} K_{g(t)} = \Delta_{g(t)} K_{g(t)} + 2K_{g(t)} \left( K_{g(t)} - \kappa \right)$$

This looks like a heat equation with one extra non-linear term, this type of PDE is called a reaction diffusion equation. Similarly to the heat equation, these equation satisfy a maximum principle property :

**Theorem 2.5.1** (Maximum Principle for reaction diffusion equations)**.** *Let $M^2$ be a compact surface wtih a time dependent Riemannian metric $g(t)$, $F : \mathbb{R} \to \mathbb{R}$ be a smooth function and $X$ be a smooth (possibly time dependent) vector field on $M^2$. Let $u : [0,T) \times M^2 \to \mathbb{R}$ be a smooth funtion such that :*

- $\dfrac{\partial}{\partial t} u \geq \Delta_{g(t)} u + \left\langle X, \nabla^{g(t)} u \right\rangle + F(u),$
- $u(0,\cdot) \geq U_0.$

*Let $U : [0,T') \to \mathbb{R}$ be such that $U' = F(U)$ and $U(0) = U_0$.*
*Then $u(t,\cdot) \geq U(t)$ for $t \in [0,T) \cap [0,T')$.*

*Proof.* Fix $\tau = \sup\{t \in [0,T) \cap [0,T')|u(x,t) > U(t\})$.

Let us compute :

$$\frac{\partial}{\partial t}(u-U) \geq \Delta_{g(t)}(u-U) + \big\langle X, \nabla(u-U)\big\rangle + F(u) - F(U).$$

Now since $F$ is smooth and $U$ and $u$ are bounded, we can find $C$ such that $|F(u)-F(U)| \leq C|u-U|$. Set $H = e^{Ct}(u-U)$, then :

$$\frac{\partial}{\partial t}H \geq \Delta_{g(t)}H + \big\langle X, \nabla H\big\rangle + F(u) - F(U) + C(u-U) \geq \Delta_{g(t)}H + \big\langle X, \nabla H\big\rangle.$$

Now the usual maximum principle for supersolutions of the heat equation applies and tells us that $H \geq 0$. Let us reproduce the proof here for completeness.

We know that at $t = 0$, $H \geq 0$. Consider $H_\varepsilon(t,x) = H(t,x) + \varepsilon + t\varepsilon$. Note that :

$$\frac{\partial}{\partial t}H_\varepsilon \geq \Delta_{g(t)}H\varepsilon + \big\langle X, \nabla H_\varepsilon\big\rangle + \varepsilon.$$

We will prove that for any $\varepsilon > 0$, $H_\varepsilon \geq 0$. Assume this is false, then there exist a smallest $t_0 > 0$ such that there exist $x_0$ with $H_\varepsilon(t_0,x_0) = 0$. At this particular point, we have :

- $\frac{\partial}{\partial t}H_\varepsilon(t_0,x_0) \leq 0$,
- $\Delta_{g(t)}H_\varepsilon(t_0,x_0) \geq 0$,
- $\nabla H_\varepsilon(t_0,x_0) = 0$.

Putting these together with the parbolic equation satisfied by $H_\varepsilon$, we get :

$$0 \geq 0 + 0 + \varepsilon > 0.$$

A contradiction. □

This tells us we will be able to control $K_{g(t)}$ in terms of the solutions to the ODE $y' = 2y(y-\kappa)$. A quick inspection of this ODE shows that :

- if $\kappa < 0$, any solution with $y(0) < 0$ will converge exponentially fast to $\kappa$, while any solution with $y(0) > 0$ will blow up in finite time.
- if $\kappa = 0$, a solution with $y(0) < 0$ will converge to 0 like $-1/t$ while a solution with $y(0) > 0$ will blow up in finite time.
- if $\kappa > 0$, any solution with $y(0) < \kappa$ will converge exponentially fast to 0, while any solution with $y(0) > \kappa$ will blow up in finite time.

After solving the equation explicitly, we get :

**Proposition 2.5.2.** *If $g(t)$ is solution to the Ricci flow on a compact surface such that at $t = 0$, $K_{g(0)} \geq \underline{K}$, then :*

- *if $\kappa < 0$ :*

$$K_{g(t)} - \kappa \geq (\underline{K} - \kappa)e^{2\kappa t}$$

- *if* $\kappa = 0$ *:*

$$K_{g(t)} \geq -\frac{1}{2t}$$

- *if* $\kappa > 0$ *:*

$$K_{g(t)} \geq \underline{K} e^{-2\kappa t}.$$

This results already gives us a hint to the fact that when $\kappa \leq 0$, the flow is quite well behaved :

**Corollary 2.5.3.** *If* $\kappa \leq 0$ *and* $g(t)$ *exists for all time* $t > 0$*, then :*

$$\int_M |K_{g(t)} - \kappa| dv_{g(t)} \xrightarrow{t \to +\infty} 0.$$

*Proof.*

$$\begin{aligned}
\int_{M^2} |K_{g(t)} - \kappa| dv_g &= \int_{M^2} (K_{g(t)} - \kappa) + 2(K_{g(t)} - \kappa)_- dv_g \\
&= \int_{M^2} 2(K_{g(t)} - \kappa)_- dv_g \\
&\leq \int_{M^2} f(t) dv_g = \text{area}(M^2, g(t)) f(t).
\end{aligned}$$

where $f(t)$ goes to zero as $t$ goes to infinity, and is given by the previous proposition. □

# 3 The uniformization theorem

The goal of the following chapters si to prove the following result :

**Theorem 3.0.4** (Hamilton,Chow). *Let $(M^2, g_0)$ be a smooth compact surface with a Riemannian metric. Let $g(t)$ be the unique Ricci flow such that $g(0) = g_0$. Then :*

- *$g(t)$ exists for all $t \in [0, +\infty)$.*
- *As $t \to \infty$, $g(t)$ converges in every $C^k$ norm to a smooth metric $g_\infty$.*
- *The metric $g_\infty$ has constant curvature $\frac{2\pi\chi(M^2)}{\mathcal{A}(M^2, g_0)}$ and is conformal to $g_0$.*

This theorem is actually strongly linked with the uniformization theorem which Riemannian surfaces (see section 3.2), which is itself a very rich theorem with connections to many different areas of mathematics.

The theorem below was first proved by Hamilton, and he used the uniformization theorem to handle the $\chi(M^2) > 0$ case. This has been circumvented and now the Ricci flow is actually able to give an independent proof of the uniformization theorem, we will present a strategy to do this in chapter 5.

## 3.1 The uniformization of Riemann surfaces

The uniformization theorem deals with Riemann surfaces.

**Definition 3.1.1.** *A Riemann surface $X$ is a complex manifold of (complex) dimension 1, that is a Hausdorff topological space which is covered by open sets $U_i$ endowed with homeomorphims $\varphi_i : U_i \to V_i$, where $V_i \subset \mathbb{C}$ is open, and such that the change of charts $\varphi_i \circ \varphi_j^{-1}$ is holomorphic wherever defined.*

Examples of Riemann surfaces are numerous:

- Any open subset of $\mathbb{C}$ is a Riemann surface. The unit disk of $\mathbb{C}$ is usually denoted by $\mathbb{D}$. The fact that an entire bounded function is constant (Liouville Theorem) shows that $\mathbb{C}$ and $\mathbb{D}$, though homeomorphic are not equivalent as Riemann surfaces.
- $\mathbb{S}^2$ can be given a Riemann surface structure by considering it as the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where a chart round the point at infinity is given by $z \mapsto 1/z$.
- If $\Lambda$ is a lattice in $\mathbb{C}$, $\mathbb{C}/\Lambda$ is a surface, homeomorphic to a 2-torus.

Riemann introduced Riemann surfaces to solve problems about algebraic curves (set of solutions to polynomial equations $P(x, y) = 0$). The connection is made in the following way:

- First, to make things more symmetric, one considers instead of

$$\{(x,y) \in \mathbb{R}^2 \,|\, P(x,y) = 0\}$$

the set

$$\mathcal{C}(\tilde{P}) = \{(X:Y:Z) \in \mathbb{CP}^2 \,|\, \tilde{P}(X,Y,Z) = 0\}$$

where $\tilde{P}$ is the unique homogeneous polynomial such that $z^d P(\frac{x}{z}, \frac{y}{z}) = \tilde{P}(x,y,z)$ where $d$ is the degree of $P$. This amounts to add to the curve its complex points and points at infinity. $\mathcal{C}(\tilde{P})$ is now compact, it is called a plane projective curve.

- If one assume that $\mathcal{C}(\tilde{P})$ is smooth, it is a (real) dimension 2 submanifold of $\mathbb{CP}^2$, and since it is defined by an holomorphic equation, the holomorphic version of the implicit function theorem will make it a Riemann surface $X_{\tilde{P}}$, this is the Riemann surface associated with the plane projective curve $\mathcal{C}(\tilde{P})$.

- Riemann then proves that :
  - two plane projective curve $\mathcal{C}(\tilde{P}_1)$ and $\mathcal{C}(\tilde{P}_2)$ have biholomorphic associated Riemann surfaces if and only if their definig polynomials satisfy :

    $$\tilde{P}_1(q(X,Y,Z), r(X,Y,Z), s(X,Y,Z)) = \tilde{P}_2(X,Y,Z)$$

    where $p, q, r$ are homogeneous polynomials of the same degree. $\mathcal{C}(\tilde{P}_1)$ and $\mathcal{C}(\tilde{P}_2)$ are said to be birationaly equivalent.
  - Any compact Riemann surface is the Riemann surface associated to some plane projective curve $\mathcal{C}(\tilde{P})$.

  So that classification of plane projective curves up to birational equivalence is reduced to the classification of compact Riemann surfaces up to biholomorphism.

The Uniformization theorem says that there are only three simply connected Riemann surfaces : the Remann sphere, the complex plane and the unit disk. The particular case of compact surfaces (which is the one we will deal with) is stated a bit differently :

**Theorem 3.1.2** (Uniformization of compact Riemann surfaces, Poincaré, Koebe)**.** *Let $X$ be a compact Riemann surface, then the universal cover of $X$ is biholomorphic to either :*

- *the Riemann sphere $\mathbb{CP}^1$,*
- *the complex plane $\mathbb{C}$,*
- *the unit disk $\mathbb{D}$.*

In each of the three cases, this tells us that $X$ can be holomorphically "parametrized" (with repetitions given by the action of the fundamental group of $X$ on $\tilde{X}$ by biholomorphisms) by one complex variable $z$ belonging to a particularly simple domain.

*Example* 3.1.3. Let us consider the smooth plane projective curve $\tilde{P}(X,Y,Z) = X^2+Y^2+Z^2 = 0$. It is a plane projective conic and after a linear coordinate change, every plane projective conic is of this form. Consider :

$$z \in \mathbb{CP}^1 \mapsto \big(1 - z^2 : 2z : i(1+z^2)\big) \in \mathbb{CP}^2$$

One can show that it is a biholomorphism from $\mathbb{CP}^1$ to $\mathcal{C}(\tilde{P})$.

The Uniformization theorem grew out of a generalization of the situation above to all plane projective curves, but at a great cost :

- apart from $\mathbb{CP}^1$, the domain can also be $\mathbb{C}$ or $\mathbb{D}$.
- the parametrization we get is only a local biholomorphism.
- the parametrization is not by polynomials anymore, but by holomorphic functions. Already to explicitly handle degree three curves, one need to introduce elliptic functions, and the case of arbitrary degree will require Fuchsian functions.

## 3.2 From Riemann surfaces to surfaces with a Riemannian metric

The Riemannian version of the Uniformization theorem for compact surfaces is usually stated in this way :

**Theorem 3.2.1.** *Let $(M^2, g)$ be a compact surface with a Riemannian metric, then one can find a smooth function $u : M^2 \to \mathbb{R}$ such that $(M^2, e^{2u}g)$ has constant curvature.*

To make the link with Uniformization of Riemann surfaces, we first remark that each of the Riemann surfaces $\mathbb{CP}^1$, $\mathbb{C}$ and $\mathbb{D}$ carry complete metrics of constant curvature :

- On $\mathbb{C}$ the euclidean metric $dx^2 + dy^2 = |dz|^2$ has curvature 0.
- On $\mathbb{D}$, the poincaré metric $\dfrac{|dz|^2}{(1-|z|^2)^2}$ has curvature $-1$.
- The Riemann sphere, with the metric $\dfrac{|dz|^2}{(1+|z|^2)^2}$ has constant curvature 1. (This the image of the standard metric on $\mathbb{S}^2$ by stereographic projection.)

Moreover :

- those three models are (up to scaling) the only simply connected surfaces of constant curvature.
- any isometry of these models is actually a biholomorphism of the underlying Riemann surface.

Now let us consider a compact Riemann surface $X$. A Riemannian metric $g$ on $X$ is said to be compatible with the complex structure if for any holomorphic chart $\varphi : U \subset \mathbb{C} \to V \subset X$, $\varphi^* g = e^{2u}|dz|^2$.

Read in charts, any local isometry between Riemann surfaces equipped with metrics co

To prove the Riemann surface Uniformization from the Riemannian Uniformization, one proceeds as follows :

- Using a partition of unity on $X$ relative to holomorphic coordinate charts, build on $X$ a Riemannian metric $g$ compatible with the complex structure.

- Then consider $\tilde{g} = e^{2u} g$ with constant curvature (whose existence is given by Riemannian uniformization).
- On the universal cover $\bar{X} \to X$, the pullback of $\tilde{g}$ will be a complete constant curvature metric, compatible with the Riemann surface structure of $\bar{X}$. With this metric $\bar{X}$ will then be isometric to either the euclidean plane, the hyperbolic plane or the round sphere, and thus biholomorphic to either $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{CP}^1$.

# 4 Convergence when $\chi(M) \leq 0$.

We have seen how to get a good lower bound on the curvature when $\kappa \leq 0$, we will now prove an upper bound which will imply convergence.

## 4.1 The potential of the curvature

The function $K_{g(t)} - \kappa$ satifies :

$$\int_{M^2} (K_{g(t)} - \kappa) dv_{g(t)} = 0.$$

So on a compact surface, we can solve the equation :

$$\Delta_{g(t)} f = K_{g(t)} - \kappa$$

moreover the solution is unique up to the addition of a suitable constant.

Now set $f(t,x)$ to be a potential of the curvature, and assume it is smooth in both variables $t$ and $x$. Now differentiate the relation $\Delta_{g(t)} f = K_{g(t)} - \kappa$ with repect to time :

$$\Delta_{g(t)} \left( \frac{\partial f}{\partial t} \right) + 2(K_{g(t)} - \kappa) \Delta_{g(t)} f = \Delta_{g(t)} K_{g(t)} + 2K_{g(t)}(K_{g(t)} - \kappa)$$

Using that $\Delta_{g(t)} f = K_{g(t)} - \kappa$, we get :

$$\Delta_{g(t)} \left( \frac{\partial f}{\partial t} \right) + 2K_{g(t)}(K_{g(t)} - \kappa) - 2\kappa \Delta_{g(t)} f = \Delta_{g(t)} K_{g(t)} + 2K_{g(t)}(K_{g(t)} - \kappa)$$

Finally, since $\Delta_{g(t)} K_{g(t)} = \Delta_{g(t)} \Delta_{g(t)} f$ :

$$\Delta_{g(t)} \left( \frac{\partial f}{\partial t} - \Delta_{g(t)} f - 2\kappa f \right) = 0.$$

Using this computation, we can show :

**Proposition 4.1.1.** *Let $f : [0, T) \times M^2 \to \mathbb{R}$ be such that*

- $f_0 = f(0, \cdot)$ *satisfies* $\Delta_{g(0)} f = K_{g(0)} - \kappa$.
- $\frac{\partial f}{\partial t} = \Delta_{g(t)} f + 2\kappa f$.

*Then for every $t > 0$, $\Delta_{g(t)} f(t, \cdot) = K_{g(t)} - \kappa$*

We will now take $f$ as in the proposition.

Considering the evolution equation for $f$, the maximum principle implies that there exists $C$ such that $|f(t,x)| \leq Ce^{\kappa t}$, where $C = \sup_{x \in M} |f_0(x)|$.

## 4.2 Using the potential to get upper bounds on the curvature.

Set $H = K_{g(t)} - \kappa + 2|\nabla f|^2$. This expression may seem arbitrary, but it has nice link to Ricci solitons which is explained in Chow and Knopff book, chapter 5 section 3.

$H$ satisfies a useful evolution equation :

**Proposition 4.2.1.**

$$\frac{\partial}{\partial t} H \leq \Delta_{g(t)} H + 2\kappa H$$

*Proof.* We use two results to make this computation :

- The fact that along a solution to the Normalized Ricci flow :

  $$\frac{\partial}{\partial t}|\nabla^g f|_g^2 = 2g\left(\nabla^g \frac{\partial f}{\partial t}, \nabla^g f\right) + 2(K_g - \kappa)|\nabla^g f|_g^2$$

  which can be proved by differentiating the identity $df \wedge * df = |\nabla^g f|_g^2 dv_g$ with respect to time.

- The Bochner formula :

  $$\tfrac{1}{2}\Delta|\nabla f|^2 = g(\nabla \Delta f, \nabla f) + |\operatorname{Hess} f|^2 + \operatorname{Ric}(\nabla f, \nabla f)$$

  together with the fact that $\operatorname{Ric}_g = K_g g$ in dimension 2. The Bochner formula could be proved using the calculus we have developped but is easier to prove using the Levi-Civitta connection formalism.

We first compute :

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(K_{g(t)} - \kappa) = 2K_{g(t)}(K_{g(t)} - \kappa) = 2\left(\Delta_{g(t)} f\right)^2 + 2\kappa(K_{g(t)} - \kappa).$$

And :

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)|\nabla f|^2 &= 2(K_{g(t)} - \kappa)|\nabla f|^2 + 2g\left(\nabla \frac{\partial}{\partial t} f, \nabla f\right) \\ &\quad - 2\left(g(\nabla \Delta f, \nabla f) + |\operatorname{Hess} f|^2 + K_{g(t)}|\nabla f|^2\right) \\ &= -2\kappa|\nabla f|^2 + 2g\left(\nabla \Delta f + 2\kappa \nabla f, \nabla f\right) \\ &\quad - 2\left(g(\nabla \Delta f, \nabla f) + |\operatorname{Hess} f|^2\right) \\ &= 2\kappa|\nabla f|^2 - 2|\operatorname{Hess} f|^2.\end{aligned}$$

Hence :

$$\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) H = 2\kappa H + 2\big((\Delta_{g(t)} f)^2 - 2|\operatorname{Hess} f|^2\big).$$

Now $\Delta_{g(t)} f$ is the trace of the Hessian, if we denote by $\lambda_1$ and $\lambda_2$ the eigenvalues of the Hessian, we get :

$$(\Delta_{g(t)} f)^2 - 2|\operatorname{Hess} f|^2 = (\lambda_1 + \lambda_2)^2 - 2(\lambda_1^2 + \lambda_2^2) = -(\lambda_1 - \lambda_2)^2 \leq 0.$$

□

**Corollary 4.2.2.** *For any solution of the normalized Ricci flow $(M^2, g(t))$, there exist $C > 0$, depending only on $g(0)$, such that :*

$$K_{g(t)} - \kappa \leq Ce^{2\kappa t}$$

*Proof.* We can apply the maximum principle to $H$ to get that :

$$H(t,x) \leq \big(\sup_{x\in M} H(0,x)\big)e^{2\kappa t}.$$

Since $K_{g(t)} - \kappa \leq H$, we have $K_{g(t)} - \kappa \leq Ce^{2\kappa t}$. □

This allows us to show long time existence of the normalized Ricci flow.

**Corollary 4.2.3.** *Let $(M^2, g_0)$ be a Riemannian surface, then the solution $g(t)$ to the normalized Ricci flow exists for $t \in [0, +\infty)$.*

*Proof.* Independently of the sign of $\kappa$, the last proposition together with the estimate from the end of chapter 2 will imply that there exist a continuous function $C : [0, +\infty) \to \mathbb{R}$ such that :

$$|K_{g(t)}| \leq C(t)$$

for any solution $g(t)$ to the normalized Ricci flow starting at $g_0$ This tells us that the curvature cannot become unbounded in finite time, the discussion in section 2.4 implies that $g(t)$ can be extended for all $t < +\infty$. □

## 4.3 Exponential convergence when $\chi(M) < 0$.

So far we have the foloowing estimate when $\kappa < 0$ :

$$|K_{g(t)} - \kappa| \leq Ce^{\kappa t}.$$

**Theorem 4.3.1.** *Let $(M^2, g_0)$ be a surface with $\chi(M^2) < 0$, and $(g(t))_{t\in[0,+\infty)}$ be the Ricci flow such that $g(0) = g_0$. Then :*

$$g(t) \xrightarrow{t\to\infty} g_\infty$$

*and $g_\infty$ is a smooth metric with constant curvature $\kappa$, conformal to $g_0$. Moreover, the convergence is exponential in every $C^m$ norm.*

We will prove the $C^0$ convergence part of the theroem in full details, the proof of $C^m$ convergence requires some more estimates and will be only sketched.

*$C^0$-convergence.* Let $V$ be a vector (time independent), then :

$$\frac{\partial}{\partial t}\big(g(t)(V,V)\big) = -2(K_{g(t)} - \kappa)g(t)(V,V)$$

So :

$$\log\left(\frac{g(t)(V,V)}{g(0)(V,V)}\right) = -2\int_0^t (K_{g(s)} - \kappa)ds$$

By our estimate $|K_{g(t)} - \kappa| < Ce^{\kappa t}$, so the function in the integral is integrable, and this is uniform in $x$, so we get uniform convergence of the metric. □

The convergence of derivatives of $g$ relies on the following estimate :

**Lemma 4.3.2.** *Let $(M^2, g(t))$ be a Ricci flow on a surface with $\chi(M^2) < 0$. Then for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that :*

$$|D^m K_{g(t)}|^2 \leq C_m e^{\kappa t}.$$

*Remark* 4.3.3. $D^m K$ denotes the $m$-th covariant derivative of the curvature.

*m=1.* We only prove the $m = 1$ case, the other ones take the same overall approach with slightly more involved computations.

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) |\nabla K_g|^2 &= 2(K_{g(t)} - \kappa)|\nabla K_{g(t)}|^2 + 2g\left(\nabla \frac{\partial}{\partial t} K_{g(t)}, \nabla K_{g(t)}\right) \\ &\quad - 2\left(g(\nabla \Delta K_{g(t)}, \nabla K_{g(t)}) + |\operatorname{Hess} K_{g(t)}|^2 + K_{g(t)}|\nabla K_{g(t)}|^2\right) \\ &= 2(K_{g(t)} - \kappa)|\nabla K_{g(t)}|^2 + 2g\left(\nabla\Big(\Delta K_{g(t)} + 2K_{g(t)}(K_{g(t)} - \kappa)\Big), \nabla K_{g(t)}\right) \\ &\quad - 2\left(g(\nabla \Delta K_{g(t)}, \nabla K_{g(t)}) + |\operatorname{Hess} K_{g(t)}|^2 + K_{g(t)}|\nabla K_{g(t)}|^2\right) \\ &= \; - 2\kappa|\nabla K_{g(t)}|^2 + 4g\left(\nabla\Big(K_{g(t)}(K_{g(t)} - \kappa)\Big), \nabla K_{g(t)}\right) \\ &\quad - 2|\operatorname{Hess} K_{g(t)}|^2 \\ &\leq (8K_{g(t)} - 6\kappa)|\nabla K_{g(t)}|^2.\end{aligned}$$

Since $K_{g(t)}$ uniformly converges to $\kappa < 0$, for $t > T$ big enough, $(8K_{g(t)} - 6\kappa) \leq \kappa$. Hence, we can apply the maximum principle on $[T, +\infty)$ to get that there exist $C$ such that :

$$\forall t \geq T, \quad |\nabla K_{g(t)}|^2 \leq C e^{\kappa t}.$$

Rescaling $C$ if needed, we can ensure that the estimate holds for all $t \geq 0$. ☐

*Sketch, $C^m$ convergence, constant curvature.* With the bounds from the lemma, we have that $D^m K_{g(t)}$ goes to zero exponentially fast, this allows to show that the convergence occurs in $C^m$ norms, and that we can make precise the following limit :

$$K_{g_\infty} = \lim_{t \to \infty} K_{g(t)} = \kappa.$$

☐

## 4.4 Adaptation in the $\chi(M) = 0$ case

We won't give any details here, please see Chow and Knopff book, section 6 of chapter 5. In a similar way to the case when $\chi(M^2) < 0$, the maximum principle can then be uesd to prove :

**Proposition 4.4.1.** *For every normalized Ricci flow $(M^2, g(t))$ on $M^2$ such that $\chi(M^2) = 0$, for every $m \in \mathbb{N}$ there exists $C_m > 0$ such that for every $t > 0$ :*

$$\sup_{x \in M^2} |D^m K_{g(t)}|^2 \leq \frac{C_m}{t^{m+2}}.$$

Using Sobolev embeddings for instance[1], this will imply that for any $n \in \mathbb{N}$ and any $m \in \mathbb{N}$ :

$$\sup_{x \in M^2} |D^m K_{g(t)}| = o(t^{-m})$$

when $t \to \infty$. Once one has these bounds, one can argue as in the case $\chi(M^2) < 0$.

[1]this is actually quite subtle, see for instance Hamilton's paper *The Ricci flow on surfaces.*

# 5 Intuition when $\chi(M) > 0$.

The situation when $\chi(M^2) > 0$ is a lot more complicated. If we look at what we have so far, the maximum principle only gave us :

$$-Ce^{-2\kappa t} \leq K_{g(t)} \leq Ce^{\kappa t}$$

with $\kappa > 0$. These bounds are not enough to control the curvature as $t \to \infty$.

The hardest part is to get good upper curvature bounds. We will outline here a strategy due to Andrews and Bryan to do this. In this chapter, we will always assume that our starting surface $(M^2, g_0)$ is a sphere $\mathbb{S}^2$ and satisfies $\kappa = 1$. In other words the area of $(M^2, g_0)$ is assumed to be $4\pi$, which can always be achieved by rescaling.

## 5.1 The isoperimetric profile

The isoperimetric inequality in the plane says that for any simple closed curve $\gamma$ in the plane, the length $L$ of $\gamma$ and the area $A$ it encloses are related by :

$$L^2 \geq 4\pi A$$

with equality if and only if $\gamma$ is a circle.

To study these kind of problems on compact surfaces, it is convenient to introduce an object called the isoperimetric profile :

**Definition 5.1.1.** *The isoperimetric profile of a compact surface $(M^2, g)$ is the function $h_g : [0,1] \to \mathbb{R}$ defined by :*

$$h_g(\beta) = \inf\big\{\mathcal{L}_g(\partial\Omega) \;\big|\; \Omega \subset M \text{ such that } \mathcal{A}_g(\Omega) = \beta\mathcal{A}_g(M)\big\}$$

*Remark* 5.1.2. When the isoperimetric profile is known, we get an isoperimetric inequality for free : for any domain $\Omega \subset M^2$ :

$$\mathcal{L}_g(\partial\Omega) \geq h_g\left(\frac{\mathcal{A}_g(\Omega)}{\mathcal{A}_g(M)}\right).$$

*Exercise* 5.1.3. Show that $h(1-\beta) = h(\beta)$.

*Exercise* 5.1.4. Show that $h_{\lambda g} = \sqrt{\lambda} h_g$.

Let $\bar{g}$ be the standard metric on the sphere of radius 1. Then a variation of the isoperimetric inequality can be proved, and the optimal curves are parallels (curves of constant lattitude on earth). With this we can prove :

$$h_{\bar{g}}(\beta) = 4\pi\sqrt{\beta(1-\beta)}.$$

Let us now consider a metric on the sphere whose area is one, and which is built by slicing a round sphere of curvature $K >> 1$ along the equator and inserting in the middle a thin long cylinder. One can then show that :

$$h(\beta) = \begin{cases} 4\pi\sqrt{\beta(1-K\beta)} & \text{for } \beta \in [0, 1/2K] \\ \frac{2\pi}{\sqrt{K}} & \text{for } t \in [1/2K, 1-1/2K]. \\ 4\pi\sqrt{(1-\beta)(1-K(1-\beta))} & \text{for } t \in [1-1/2K, 1] \end{cases}$$

The reason is the following :

- when $4\pi\beta$ is smaller than the area of an hemisphere of curvature $K$, the regions of area $4\pi\beta$ with minimal boundary length are inside the extremal hemispheres.
- after that, the optimal domains all have the same length : their bounday is always a circle inside the cylindrical part.

## 5.2 Upper bounds on the curvature from isoperimetric comparison.

The previous example demonstrates the fact that for $\beta << 1$, domains $\Omega \subset$ of area $4\pi\beta$ which will satisfy $\mathcal{L}(\partial\Omega) = h(\beta)$ will tend to concentrate around the points where the curvature is the biggest. The following statement makes this intuition precise :

**Proposition 5.2.1.** *Let $(M^2, g)$ be a surface of area $4\pi$. Then :*

$$h_g(\beta) = 4\pi\sqrt{\beta}\left(1 - \frac{\sup_{x\in M} K_g(x)}{2}\beta + O(\beta^{3/2})\right)$$

*as $t$ goes to 0.*

*Remark* 5.2.2. This tells us that at really small scales, the euclidean isoperimetric inequality is satisfied up to an error term wich depends only on the maximum value of the curvature.

**Corollary 5.2.3.** *Let $(M, g)$ and $(\tilde{M}, \tilde{g})$ be two Riemannian surfaces whose isoperimetric profile satisfy : $h \leq \tilde{h}$. Then $\sup_M K_g \geq \sup_{\tilde{M}} K_{\tilde{g}}$.*

## 5.3 Bounding the isoperimetric profile under the Ricci flow.

We will consider a special kind of metrics which we will call (by lack of better name) a sausage metric.

**Definition 5.3.1.** *A sausage metric is a metric $g$ on $\mathbb{S}^2$ such that :*

- *$g$ is rotationnally symmetric,*
- *$g$ admits a symmetry with respect to an "equator",*
- *The curvature $K_g$ is striclty positive and decreases as one goes from the tip to the equator.*

*Remark* 5.3.2. More formally, the metric can be written as $g = dr^2 + f(r)^2 d\theta^2$ for $r \in (0, \pi)$ and $\theta \in (0, 2\pi)$ (this ensures rotiational symmetry), and $f$ satisfies :

- $f(r) = r + o(r)$ and $f(\pi - r) = r$ when $r \to 0$, this ensure that this defines a metric on $\mathbb{S}^2$.
- $f(\pi - r) = f(r)$, this give the symmetry with respect to the equator $r = \pi/2$.
- $r \mapsto -\frac{f''}{f}(r)$ is strictly positive and decreasing for $r \in (0, \pi/2)$, this gives the required positivity and monotonicity of the curvature.

The isoperimetric profile of sausage metrics can be finely studied and its evolution under the Ricci flow can be controlled.

Andrews and Bryan prove the following result :

**Theorem 5.3.3.** *Let $(\mathbb{S}^2, g(t))$ and $(\mathbb{S}^2, \bar{g}(t))$ be two normalized Ricci flows such that :*

- *$\bar{g}(t)$ is a sausage metric for all $t$.*
- *$h_{g(0)}(\beta) \geq h_{\bar{g}(0)}(\beta)$ for all $\beta \in (0, 1)$.*

*Then for all $t > 0$, $h_{g(t)} \geq h_{\bar{g}(t)}$.*

## 5.4 Comparison with the Rosenau solution.

The Rosenau solution $(\mathbb{S}^2, g_R(t))$ is one of the few explicit solutions to the Normalized Ricci flow. It is a solution on the sphere which as the following properties :

- The area of $g_R(t)$ is constant equal to $4\pi$.
- for all $t$, $g_R(t)$ is a sausage metric.
- when $t \to +\infty$, $K_{g_R(t)}$ uniformly converges to 1. The convergence is exponentialy fast.
- as $t$ goes to $-\infty$, $g_R$ looks more and more like a long thin cylinder with closed by two rounds caps where most of the curvature is concentrated. In particular :

$$\max_{\mathbb{S}^2} K_{g_R(t)} \xrightarrow{t \to -\infty} +\infty$$

  and $h_{g_R(t)}$ goes to 0 uniformly as $t \to -\infty$.

With this at hand we can prove :

**Theorem 5.4.1.** *Let $(\mathbb{S}^2, g(t))$ be a solution to normalized Ricci flow. Then there exist $C > 0$ and $\alpha > 0$ such that :*

$$K_{g(t)} - 1 \leq Ce^{-\alpha t}.$$

*Proof.* Let $h_{g(0)}$ be the isoperimetric profile of $g(0)$. The properties of the Rosenau solution described above imply that there is a time $-t_0$ such that $h_{g_R(-t_0)} \leq h_{g(0)}$.

Thus for every $t > 0$, Theorem 5.3.3 implies :

$$h_{g_R(t-t_0)} \leq h_{g(t)}.$$

And now, Corrolary 5.2.3 implies :

$$K_{g(t)} \leq K_{g_R(t-t_0)}.$$

Since $K_{g_R(t-t_0)}$ goes to 1 exponentialy fast, the theorem is proved. $\square$

From this we can show that :

$$\int_{\mathbb{S}^2} |K_{g(t)} - 1| dv_{g(t)} \xrightarrow{t \to \infty} 0.$$

With some extra analytic work which is presented in the paper by Andrews an Bryan, one can show that :

$$\max_{\mathbb{S}^2} |K_{g(t)} - 1| \leq C' e^{-\alpha t/2}.$$

The convergence to a constant curvature metric then follows as in chapter 4.